

A Level Mathematics

Matrices

Autumn First Half

This booklet accompanies your course textbook.

It contains the key skills, worked examples, consolidation questions,
practice questions and exam-style questions for this chapter.

Name: _____

Class: _____

Chapter Overview

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Key Vocabulary and notes

KWL

Know	Want to learn	Learnt

Key Words

Functions Domain Range Composite Inverse

Key Notes

Fill in the blanks from the word bank below.

Word Bank

word1 word2

How does arithmetic with matrices differ from arithmetic with numbers?

Learning Objective

TBAT identify the dimensions of a matrix

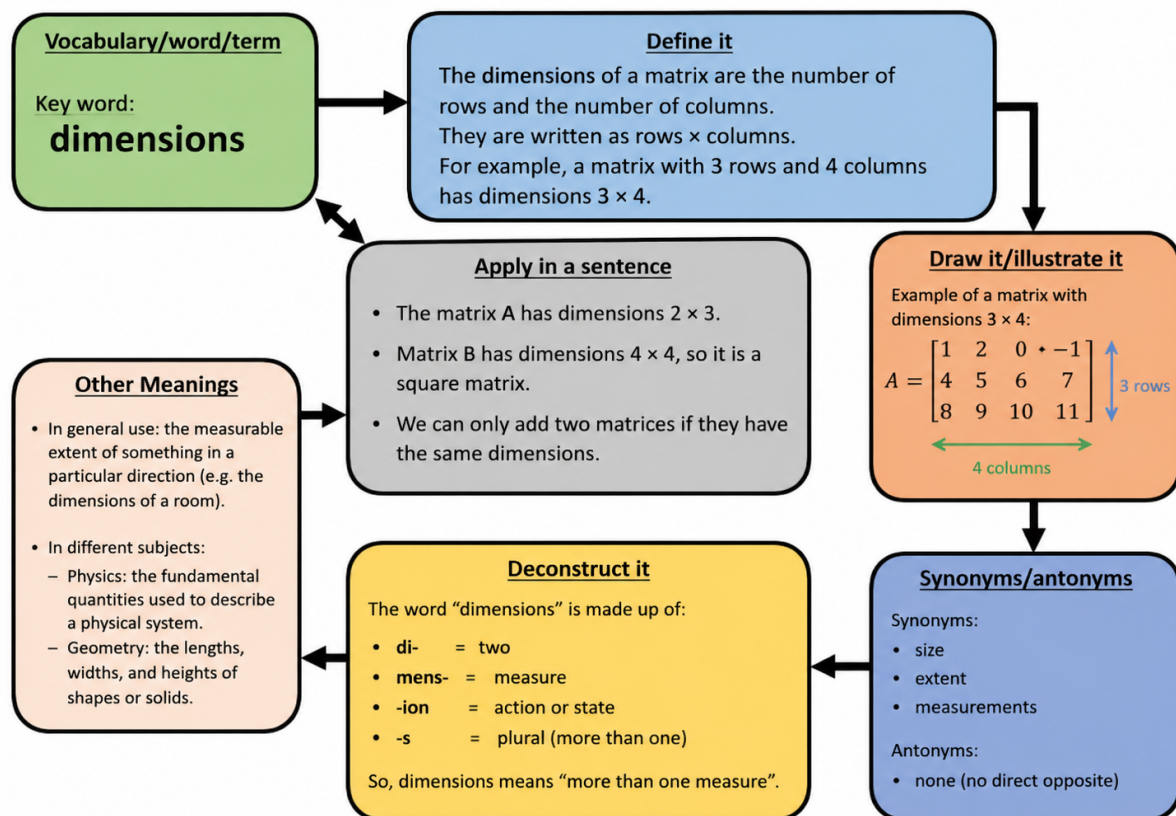
KEY POINT

A matrix is a rectangular grid or array of numbers, symbols, or expressions arranged in horizontal rows and vertical columns.

KEY POINT

A vector is a simple example of a matrix

Frayer model



KEY POINT

The dimensions of a matrix is written as:

Number of rows $\times$ Number of columns

I DO

What are the dimensions of the following matrices.

$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 5 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 7 & 8 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$$

YOU DO

What are the dimensions of the following matrices.

$$\begin{pmatrix} 1 & 2 & 2 & 6 \\ 3 & 5 & 5 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 5 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 4 \\ 6 & 7 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 7 & 8 \end{pmatrix}$$

KEY POINT

If the number of rows and number of columns are equal, it is called a **square matrix**:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

KEY POINT

If all the entries in a matrix are zero, this is called a **zero matrix**. This is denoted by **0**.

KEY POINT

An identity matrix is a **square matrix** with all 0 apart from a diagonal line of 1s. An identity matrix is denoted by I_n where n is the dimension of the matrix. **square matrix**:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Textbook practice

For further practice, complete the relevant questions from:

Edexcel AS Further Maths - Core Pure Mathematics

Exercise 6A, Q1-3

Learning Objective

TBAT add and subtract matrices

KEY POINT

You can only **add** and **subtract** matrices if they are the same size, their dimensions are the same

I DO

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -2 \\ 3 & -5 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix},$$

Find $A + B$

Find $B - C$

Find $C + D$

YOU DO

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -2 \\ 3 & -5 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix},$$

Find $A + A$

Find $B - D$

Find $D + D$

Independent practice

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 4 & 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -2 \\ -3 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 4 & 1 \\ 1 & -4 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}, \quad E = I_2$$

Question 1: List all possible pairs of matrices that you can add from the matrices above.

Question 2: Find $B + E$ and find $E + B$. What do you notice about the answers?

Question 3: Find $A + A + A$

Find $B + B$

Find $E - B$

Textbook practice

For further practice, complete the relevant questions from:

Edexcel AS Further Maths - Core Pure Mathematics

Exercise 6A, Q4-9

Learning Objective

TBAT multiply matrices by scalars

I DO

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -2 \\ 3 & -5 \\ 5 & 2 \end{pmatrix},$$

Find 3A

Find -2B

Find 2A + C

YOU DO

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -2 \\ 3 & -5 \\ 5 & 2 \end{pmatrix},$$

Find -2A

Find 3B

Find 5A - 2C

Textbook practice

For further practice, complete the relevant questions from:

Edexcel AS Further Maths - Core Pure Mathematics

Exercise 6A, ALL

Learning Objective

TBAT multiply two matrices

KEY POINT

The dot product is when you multiply two column vectors of the same size by multiplying the first component of the first vector by the first component of the second vector. Then multiply the second component of the first vector by the second component of the second vector. Repeat this for each element and then sum your answers.

$$\begin{pmatrix} a_1 \\ a_2 \\ a_{n-1} \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_{n-1} \\ b_n \end{pmatrix} \\ = a_1 \times b_1 + a_2 \times b_2 + a_{n-1} \times b_{n-1} + a_n \times b_n$$

I DO

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$$

Workout the following dot products

YOU DO

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ -5 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \\ -6 \\ 8 \end{bmatrix}$$

Workout the following dot products

KEY POINT

Matrices can be multiplied together if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

KEY POINT

If matrix A has dimensions $(n \times m)$ and matrix B has dimensions $(m \times k)$ then matrix AB has dimension $(n \times k)$

KEY POINT

In general $AB \neq BA$ even when both matrices are square matrices.

I DO

$$A = \begin{pmatrix} 1 & 2 & 2 & 6 \\ 3 & 5 & 5 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 4 \\ 6 & 7 & 8 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 7 & 8 \end{pmatrix}$$

Find the dimensions of the following matrices.

Hence, state whether it is possible to complete the following multiplications. If possible state the dimensions.

$$AB \quad BA \quad DE \quad ED$$

YOU DO

$$CA \quad CB \quad CC \quad CD \quad AC \quad BC \quad DC$$

KEY POINT

To multiply two matrices AB , you find the dot product of the corresponding row of matrix A by the corresponding column of matrix B .

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \left(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cdot \begin{pmatrix} e \\ g \\ e \\ g \end{pmatrix} \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cdot \begin{pmatrix} f \\ h \\ f \\ h \end{pmatrix} \right)$$

I DO

$$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}, \quad D = I_2$$

If possible find AB

If possible find BA

If possible find AC

If possible find CD

YOU DO

$$A = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad D = I_2$$

If possible find AB

If possible find BA

If possible find BC

If possible find CD

Textbook practice

For further practice, complete the relevant questions from:

Edexcel AS Further Maths - Core Pure Mathematics

Exercise 6B, ALL

What decides whether a matrix's effect can be reversed, and how do we reverse it?

Learning Objective

TBAT calculate the determinant of a 2×2 matrix

KEY POINT

The determinant is a scalar value associated with a square matrix.

Key Notes

Geometric Meaning

Scaling factor: The absolute value of the determinant tells you the factor by which area (in 2D) or volume (in 3D and higher) changes when you apply a matrix transformation.

Orientation: The sign of the determinant shows if the space keeps its original orientation (positive) or flips like a mirror image (negative).

Collapse: A determinant of zero means the space collapses completely into a lower dimension, such as a 3D box or 2D square squished flat into a line or point.

KEY POINT

If the determinant is 0, that means the matrix is a singular matrix.

Singular matrices do not have an inverse.

KEY POINT

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

I DO

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Find $\det A$

YOU DO

$$A = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix}$$

Find $\det A$

Independent practice

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}, \quad E = \begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}, \quad F = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

Find $\det A$

What is the difference between matrix A and B
Find $\det B$

What is the difference between matrix A and C
Find $\det C$

What is the difference between matrix A and D
Find $\det D$

What is the difference between row 1 of matrix A and E
Find $\det E$

What is the difference between matrix A and F
Find $\det F$

Textbook practice

For further practice, complete the relevant questions from:

Edexcel AS Further Maths - Core Pure Mathematics

Exercise 6C, Q1 - Q5

Learning Objective

TBAT calculate the determinant of a 3×3 matrix

KEY POINT

To find the determinant of a 3×3 matrix, you select a row or column and multiply each element in that row or column by its minor and cofactor and then add the results.

KEY POINT

The cofactor matrix:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

KEY POINT

The minor of an element in a matrix is when you remove the row and column that contains the element and find the determinant of what remains.

Minor of element 'a':

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} e & f \\ h & i \end{pmatrix} \rightarrow \begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - fh$$

Minor of element 'b':

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} d & f \\ g & i \end{pmatrix} \rightarrow \begin{vmatrix} d & f \\ g & i \end{vmatrix} = di - fg$$

Minor of element 'c':

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} d & e \\ g & h \end{pmatrix} \rightarrow \begin{vmatrix} d & e \\ g & h \end{vmatrix} = dh - eg$$

KEY POINT

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (+) a \begin{vmatrix} e & f \\ h & i \end{vmatrix} (-) b \begin{vmatrix} d & f \\ g & i \end{vmatrix} (+) c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-) d \begin{vmatrix} b & c \\ h & i \end{vmatrix} (+) e \begin{vmatrix} a & c \\ g & i \end{vmatrix} (-) f \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (+) g \begin{vmatrix} b & c \\ e & f \end{vmatrix} (-) h \begin{vmatrix} a & c \\ d & f \end{vmatrix} (+) i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

I DO

Find:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$$

Find:

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} =$$

YOU DO

Find:

$$\begin{vmatrix} 3 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 2 \end{vmatrix} =$$

Find:

$$\begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{vmatrix} =$$

Independent practice

Calculate the determinant of the following matrix in 2 different ways.

$$\begin{vmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 8 & 9 \end{vmatrix} =$$

Textbook practice

For further practice, complete the relevant questions from:

Edexcel AS Further Maths - Core Pure Mathematics

Exercise 6C, all questions

Learning Objective

TBAT calculate the inverse of a 2×2 and 3×3 matrix

KEY POINT

Only square matrices with a non zero determinant can be inverted.

KEY POINT

5 step guide to finding the inverse of matrix A

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

:

1. Find $\det A$
2. Find the matrix of minors

$$\begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & \begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

3. Multiply each element of the matrix of minors by the sign from the matrix of cofactors.

$$\begin{pmatrix} + \begin{vmatrix} e & f \\ h & i \end{vmatrix} & - \begin{vmatrix} d & f \\ g & i \end{vmatrix} & + \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ - \begin{vmatrix} b & c \\ h & i \end{vmatrix} & + \begin{vmatrix} a & c \\ g & i \end{vmatrix} & - \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ + \begin{vmatrix} b & c \\ e & f \end{vmatrix} & - \begin{vmatrix} a & c \\ d & f \end{vmatrix} & + \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

4. Now find the transpose of this matrix.
(The transpose of a matrix is found by interchanging rows and columns)

$$\begin{pmatrix} + \begin{vmatrix} e & f \\ h & i \end{vmatrix} & - \begin{vmatrix} d & f \\ g & i \end{vmatrix} & + \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ - \begin{vmatrix} b & c \\ h & i \end{vmatrix} & + \begin{vmatrix} a & c \\ g & i \end{vmatrix} & - \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ + \begin{vmatrix} b & c \\ e & f \end{vmatrix} & - \begin{vmatrix} a & c \\ d & f \end{vmatrix} & + \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} + \begin{vmatrix} e & f \\ h & i \end{vmatrix} & - \begin{vmatrix} b & c \\ h & i \end{vmatrix} & + \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ - \begin{vmatrix} d & f \\ g & i \end{vmatrix} & + \begin{vmatrix} a & c \\ g & i \end{vmatrix} & - \begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ + \begin{vmatrix} d & e \\ g & h \end{vmatrix} & - \begin{vmatrix} a & b \\ g & h \end{vmatrix} & + \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

5. Finally divide each element by the determinant.

$$\frac{1}{\det A} \begin{pmatrix} + \begin{vmatrix} e & f \\ h & i \end{vmatrix} & - \begin{vmatrix} b & c \\ h & i \end{vmatrix} & + \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ - \begin{vmatrix} d & f \\ g & i \end{vmatrix} & + \begin{vmatrix} a & c \\ g & i \end{vmatrix} & - \begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ + \begin{vmatrix} d & e \\ g & h \end{vmatrix} & - \begin{vmatrix} a & b \\ g & h \end{vmatrix} & + \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

KEY POINT

Short cut for 2×2 matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

I DO

Find the inverse of the following matrix:

$$\begin{pmatrix} 4 & 2 \\ 5 & 5 \end{pmatrix}$$

YOU DO

Find the inverse of the following matrix:

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$$

I DO

Find the inverse of the following matrix:

$$\begin{pmatrix} 7 & 0 & 0 \\ 3 & 4 & 9 \\ 5 & 6 & 8 \end{pmatrix}$$

YOU DO

Find the inverse of the following matrix:

$$\begin{pmatrix} 1 & 4 & 9 \\ 0 & 3 & 0 \\ 5 & 6 & 8 \end{pmatrix}$$

Textbook practice

For further practice, complete the relevant questions from:

Edexcel AS Further Maths - Core Pure Mathematics

Exercise 6D and 6E, all questions

What does it mean for a system of equations to have no solutions, or infinitely many - and how does the matrix tell us which?

What does it mean for a system of equations to have no solutions, or infinitely many - and how does the matrix tell us which?

Learning Objective

TBAT solve systems of equations using matrices

Independent practice

Workout:

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

What does it mean for a system of equations to have no solutions, or infinitely many - and how does the matrix tell us which?

KEY POINT

A set of linear equations are consistent when there is at least one set of solutions that satisfies all the equations.

KEY POINT

if

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = v$$

then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}v$$

where, A is a (3×3) matrix and v is a (3×1) matrix

I DO

Formulate a set of linear equations and then find a solution using matrices. You may use a calculator.

There are 2000 people in a school: male students, female students and teachers. There are 100 more male students to female students. After a trip left the school:

- Number of male students decreased by 10
- Number of female students decreased by 8
- Number of teachers decreased by 5
- There were now 1823 people in the school.

What does it mean for a system of equations to have no solutions, or infinitely many - and how does the matrix tell us which?

YOU DO

Formulate a set of linear equations and then find a solution using matrices. You may use a calculator.

There are 2000 people in a school: male students, female students and teachers. There are 150 more male students to female students. After a trip left the school:

- Number of male students decreased by 20
- Number of female students decreased by 10
- Number of teachers decreased by 10
- There were now 1700 people in the school.

Textbook practice

For further practice, complete the relevant questions from:

Edexcel AS Further Maths - Core Pure Mathematics

Exercise 6F, all questions